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Inverse problems for stochastic heat conduction with moving boundaries

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ABSTRACT:
The paper analyses inverse problems for stochastic heat conduction with moving boundaries due to phase change. In the case of deterministic analysis, the direct problem of such an issue is known as the Stefan problem. The inverse problems may concern the material parameters such as latent heat, thermal conductivity, specific heat which can be assumed to be random as well as stochastic characteristics of the process such as an expression for covariance.

KEYWORDS:
stochastic heat conduction; moving boundaries; inverse problem

1. Introduction

Inverse problems in mathematical physics are a class of problems focused on determining unknown causes based on observed effects, rather than predicting effects from known causes [1]. Stochastic direct heat conduction problems are analyzed, for instance, in [2-6]. In the context of heat conduction, inverse problems are crucial in applications where the thermal properties, sources, or boundary dynamics are unknown and must be deduced from observed temperature data [7-9].

The study of stochastic heat conduction with moving boundaries introduces additional layers of complexity [10]. Here, the goal is to understand heat transfer within a domain where boundaries may change over time, as seen in processes such as melting, growth of materials, or even dynamic biological environments. Stochastic elements further complicate the model by introducing random fluctuations – this is often necessary for capturing realistic conditions influenced by uncertain or fluctuating environmental factors.

This article explores inverse problems related to such stochastic heat conduction systems with moving boundaries. We aim to outline methods for identifying key parameters and boundary conditions in scenarios where both randomness and boundary dynamics are present. Approaches to this topic have broad implications, from industrial applications like metal cooling and thermal insulation to fields such as biomedical engineering, where heat conduction in dynamically changing tissues can provide valuable diagnostic information. This article presents recent advancements, methodologies, and challenges in tackling these inverse problems, emphasizing how these approaches can enhance predictive accuracy and practical applicability in complex, real-world systems.

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2. Substantive justification

The study of inverse problems in stochastic heat conduction with moving boundaries addresses a complex but essential issue in understanding and predicting heat behavior in dynamic, uncertain environments. Traditional heat conduction problems, governed by deterministic partial differential equations, assume fixed boundaries and known properties. However, in many practical applications, boundaries evolve over time, and randomness plays a significant role in material properties, environmental conditions, or internal sources, challenging standard modeling and solution methods.

At the heart of this issue is the need to extract unknown parameters – such as thermal diffusivity, heat sources, or boundary positions by working backward from observed temperature data. This process becomes exponentially more difficult when boundaries move, as in cases of melting or material expansion, and when the system is subject to random fluctuations. These conditions require a departure from conventional inverse problem-solving techniques. Instead, they necessitate advanced methods like stochastic partial differential equations (SPDEs), which incorporate probabilistic elements into the heat conduction model. SPDEs allow for a more realistic representation of how heat behaves under random influences and with shifting boundaries, but they are computationally intensive and complex to solve.

Another layer of difficulty lies in handling the uncertainties introduced by both the stochastic nature of the model and the moving boundaries, which are often highly nonlinear. For example, in processes where phase changes occur, such as melting, the boundary movement itself is influenced by temperature, creating a feedback loop that complicates analysis. This feedback effect means that any changes in estimated parameters or boundary conditions can have a ripple effect, influencing future states of the system in unpredictable ways.

From an applications standpoint, solving these inverse problems accurately has substantial implications. In industrial manufacturing, understanding and controlling heat in materials undergoing phase changes or expansions can improve quality and efficiency, as in casting and welding processes. In environmental studies, the ability to model ground temperature fluctuations with moving boundaries can inform predictions related to permafrost thawing or soil freezing. In biomedical engineering, accurately modeling tissue temperature with uncertain boundaries, such as in tumors with irregular shapes, can enhance the safety and precision of thermal therapies.

Despite recent advances in theoretical and computational methods, challenges remain in achieving stable, reliable solutions to these inverse problems. Data acquisition is often noisy, which complicates the extraction of meaningful parameters. Furthermore, developing methods that are computationally efficient yet robust to uncertainties remains a key hurdle. As such, this field continues to push the boundaries of applied mathematics, spurring the development of novel techniques in statistical modeling, numerical analysis, and machine learning to meet these challenges and extend the practical reach of stochastic heat conduction models with dynamic boundaries.

3. Problem formulation

$b(\mathbf{x},\omega)$ with $\omega \in \Omega$ is a random field at point $\mathbf{x}$ defined in a probability space (Ω, Z, P). Ω is the set of outcomes, Z is the σ-algebra of events and P is a probability measure.

Γ_t is the interface between the two phases at time t with temperature $T = T(\mathbf{x},t,\omega)$ and ν is the unit outward normal vector to the second (solid) phase. The Stefan condition determines the evolution of the surface Γ by giving an expression for velocity of the free surface in the direction ν with the latent heat $L = L(\mathbf{x},t,\omega)$.

The energy balance equation in regions considered (liquid and solid) is

$$C(\mathbf{x},\omega,T)\partial_t T + \nabla \mathbf{q}(\mathbf{x},t,\omega) + \mathrm{Q} = 0 \tag{1}$$

where C is the function of specific heat and density, $\mathbf{q}$ is the heat flux and Q is the volumetric heat flux.

The equation of Fourier is

$$\mathbf{q}(\mathbf{x},t,\omega) = -k(\mathbf{x},\omega,T)\,\nabla T(\mathbf{x},t,\omega) \tag{2}$$

where $k(\mathbf{x},\omega,T)$ is the thermal conductivity.

In the approach the phase change is modeled by a variant of the enthalpy method

$$C(\mathbf{x},\omega,T) = \mathrm{dH}/\mathrm{d}T \tag{3}$$

where $\mathrm{H} = \mathrm{H}(\mathbf{x},\omega,T)$ is the enthalpy.

The random field $b(\mathbf{x})$ can be approximated by finite elements with shape functions N_i that gives $b(x)=\sum_{i=1}^{m} N_i\, b_i = \mathbf{N}\,\mathbf{b}$ with b_i being the nodal values of b at x_i, i = 1,...,m. The mean value of the discretized vector $\mathbf{b} = \mathbf{b}(\mathbf{x})$ is $\mathrm{E}[\mathbf{b}] = \sum_{i=1}^{m} \mathrm{N}_i\,\mathrm{E}(b_i)$.

The equation (1) can be transformed to a matrix equation. By the use of the finite element method we get the equation for the temperature vector $\mathbf{T} = \mathbf{T}(\mathbf{b},t)$ with elements T_I of temperatures in nodes as

$$\mathbf{C}(\mathbf{b})\,\dot{\mathbf{T}}(\mathbf{b},t) + \mathbf{K}(\mathbf{b})\,\mathbf{T}(\mathbf{b},t) = \mathbf{F}(\mathbf{b},t) \tag{4}$$

where $\mathbf{C}$ is the heat capacity matrix, $\mathbf{K}$ is the thermal conductivity matrix, $\mathbf{F}$ is the thermal load vector and dot indicates the temperature rates.

The nonlinear stochastic equation of the form is

$$\mathbf{T}_t^{(n)} + f(\mathbf{T}_t,\ldots,\mathbf{T}_t^{(n-1)},\varepsilon) = \mathbf{Y}_t \quad t \in I \tag{5}$$

where $\varepsilon \in (0,\varepsilon^0)$ is the so-called small parameter. Assume that in time interval I the process has continuous realizations with probability equals to 1.

If f is analytic then with probability equals to 1 the solution can be expressed in the form of series of random processes

$$\mathbf{T}_t^{\varepsilon} = \mathbf{T}_t^{o} + \sum_{n=1}^{\infty} \varepsilon^n\, \mathbf{T}_t^{n} \tag{6}$$

uniformly convergent on $\mathbf{T}$ with probability 1. For sufficiently small ε the difference between solutions is not only small, but can be estimated with any accuracy.

The formula for expected values for temperature in node I for two terms is

$$\mathrm{E}\left[\mathrm{T_I}\right] = \mathrm{T_I}(\mathrm{p}^0) + \frac{1}{2}\sum_{\alpha,\beta=1}^{\mathrm{m}} \frac{\mathrm{d}^2\,\mathrm{T_I}}{\mathrm{dp}_\alpha\,\mathrm{dp}_\beta}\mathrm{Cov}(\mathrm{p}_\alpha,\mathrm{p}_\beta) \tag{7}$$

where p is any random parameter i.e. thermal conductivity, specific heat, enthalpy, etc.

And the expression for covariance of temperatures is

$$\mathrm{Cov}(\mathrm{T_I},\mathrm{T_J}) = \frac{\mathrm{dT_I}}{\mathrm{dp}_\alpha}\frac{\mathrm{dT_J}}{\mathrm{dp}_\beta}\mathrm{Cov}(\mathrm{p}_\alpha,\mathrm{p}_\beta) \tag{8}$$

where m is the number of nodes, I, J = 1,2,..., m, α, β = 1,2,..., m.

The following expression for the covariance is assumed to be

$$\mathrm{Cov}\left(p_\alpha, p_\beta\right) = \vartheta \cdot \exp\left(-\frac{\left|x_\alpha - x_\beta\right|}{\lambda}\right) \tag{9}$$

where ϑ is the correlation length and λ is the value of diagonal of covariance.

In solution of the inverse problem concerning the material characteristics the Newmark method can be used to compute equations in time. A conjugate gradient method approach can be adopted for the solution.

4. Conclusion

The article focuses on the challenging task of determining unknown parameters or boundary behaviors in heat conduction models that feature both randomness and dynamically changing boundaries. These types of problems are commonly encountered in systems where the material interface changes over time, such as in melting or freezing processes, material growth, and biomedical applications involving tissue dynamics.

In particular, the article delves into inverse problem techniques that allow researchers to estimate thermal properties, source terms, or boundary positions based on observed temperature data. The inclusion of stochastic elements adds realism to these models, as real-world systems often experience random fluctuations due to environmental or material uncertainties. This complexity requires specialized mathematical and computational tools, such as stochastic partial differential equations and probabilistic modeling approaches, to accurately capture both the randomness and the moving boundary dynamics.

The article reviews recent methods and theoretical advancements in this field, discusses the practical implications of solving such inverse problems, and highlights potential applications. These approaches have promising applications in industrial processes, environmental monitoring, and biomedical engineering, where accurate temperature modeling can provide insights into material properties, predict system behaviors, and support decision-making in dynamic environments.

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Problemy odwrotne dla stochastycznego przewodzenia ciepła z ruchomymi granicami

STRESZCZENIE:
Przeanalizowano problemy odwrotne dla stochastycznego przewodzenia ciepła z ruchomymi granicami ze względu na zmianę fazy. W przypadku analizy deterministycznej bezpośredni problem takiego zagadnienia znany jest jako problem Stefana. Problemy odwrotne mogą dotyczyć parametrów materiału, takich jak ciepło utajone, przewodnictwo cieplne, ciepło właściwe, które można uznać za losowe, a także stochastycznych charakterystyk procesu, takich jak wyrażenie na kowariancję.

SŁOWA KLUCZOWE:
stochastyczne przewodzenie ciepła; ruchome granice; problem odwrotny