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About one problem of finding equal strength contour inside a viscoelastic rectangle

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ABSTRACT:
The problem of finding an equal-strength contour inside a viscoelastic rectangle according to the Kelvin-Voigt model is considered. It is assumed that constant normal compressive forces with given principal vectors act on the sides of the rectangle (or the values of constant normal displacements are known), and the inner boundary (the desired equal-strength contour) is free from external forces. The methods of the theory of conformal reflections, Cauchy type integral and boundary value problems of analytic functions are used to study the plate bending problems discussed in the paper. Which in turn are based on the task of constructing a conformally mapping function on a doubly linked circular ring bounded by broken line. The latter is reduced to the Riemann-Hilbert problem for a circular ring based on the solution of which it becomes possible to present the mentioned function in a defective form. It is worth noting that when considering mixed problems of plate bending for doubly connected areas bounded by broken line, it is possible to decompose them into two independent problems, each of which is a Riemann-Hilbert problem.

KEYWORDS:
Kelvin-Voigt model; Kolosov-Muskhelishvili formulas; Riemann-Hilbert problems; Volterra equation

1. Introduction

The problems of the theory of plate bending belong to one of the important problems of the theory of elasticity, and the relevance of their study is due to the numerous practical applications that the plates, as construction elements, have found in construction practice, as well as in shipbuilding, machine building, aircraft construction, and others, whose mathematical foundations originate from the 20s of the 20th century. At the same time, it became the subject of research by many scientists, and this process continues today.

Among the various methods that have been developed for the calculation of plates, one of the important place takes the complex analysis methods, such as conformal reflections and the theory of boundary value problems of analytic functions [1, 2], which are systematically used in both flat theory of elasticity and plate bending problems mainly relats to the names of Kolosov, Muskhelishvili [3] and theirs numerous followers and disciples [4-10].

Among them, the contribution of Georgian scientists is particularly noteworthy, in addition, the analogues who discovered between problems of flat theory of elasticity and plate bending (in both cases, we are dealing with boundary problems of biharmonic equations), allowed scientists to use the same approach to solve this problem.

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From the point of view of practical application, one of the most important places is occupied by the problems of flat theory of elasticity and plate bending for two-link areas bounded by broken line [11, 12].

The most successful discovery in terms of efficient (analytical) construction of their solutions is the boundary conditions of analytic functions, in particular, the theory of Dirichlet and Riemann-Hilbert problems, which is essentially based on the issue of constructing a conformally mapping function on a circular ring of a given cut. If the simple link bounded by a polygon is a specific mapping function on a circle, the Christoffel-Schwartz formula is presented.

For areas bounded by polygons, both in plane of elasticity and in plate bending problems, it is essential to determine the concentration of stresses in the vicinity of the corner vertices. which in turn is reflected in the vicinity of the hole in general, determining the stress concentration is one of the important tasks in the flat theory of elasticity and plate bending problems, and they are detailed in G. Savin's famous monograph [13].

For the finite two-linked array studied by Bantsuri [1, 2, 9] and the solution of the tasks has put on the agenda such important issues of optimal design as reducing the stress concentration in the vicinity of the vertices of the planes and finding an equally strong contour inside the polygon.

Non-adjustment of geometric and physical parameters in thin-walled structures leads to a significant concentration of stresses and creates dangerous zones of plastic deformation of cracks [14-23].

In scientific innovation, it is effectively constructed Kolosov-Mushkelishvili's bonded complex broken lines, for both finite and infinite double-linked bonded areas in different cases of plate bending, leaned on jointed double-linked plate, which bends under a normal load of a certain intensity: a finite double-linked plate, the inner boundary of which is rigidly supported at the outer boundary on the defining sections.

2. Problem

Let S be a doubly connected region whose outer boundary is a rectangle A_1, A_2, A_3, A_4 whose sides are parallel to the coordinate axes, and whose inner boundary is a smooth closed contour (an unknown part of the boundary of the region S). It is assumed that normally compressive stresses with known principal vectors act on the sides of the rectangle (or constant normal displacements are given $V_n(\sigma) = \text{const}$), and the inner part (the desired equal-strength contour) is free from extrenal forces. The equal strength of the desired contour lies in the fact that the tangential normal stress acting on it at each point of the contour takes the same value depending only on time t, i.e. $\sigma_\vartheta(z,t) = K_0(t)$. The viscoelasticity of the S region as understood by the Kelvin-Voight model.

To solve the problem, methods of complex analysis are used (methods of the theory of conformal mappings and boundary value problems of analytic functions), and the equation of the desired contour is written in an analytical form.

Similar problems of the plane theory of elasticity and plate bending are considered in [1, 2, 4-9].

3. Problem solution

Let us present some results from [10] and [3]. In particular, the boundary conditions of the second main problem of the plane theory of viscoelasticity acording to the Kelvin-Voight model can be written in the form [10]

$$\int_0^t ae^* e^{k(\tau-t)}\varphi(\sigma,\tau) + \left(\varphi(\sigma,\tau) - \sigma\overline{\varphi'(\sigma,\tau)} - \overline{\psi(\sigma,\tau)}\right) 2\mu(u+iv), \sigma \in L \tag{1}$$

or

$$\int_0^t \left[ae^* e^{k(\tau-t)} + 2e^{m(\tau-t)}\right]\Phi(\sigma,\tau)d\tau - \int_0^t e^{m(\tau-r)}\left(\Phi(\sigma,\tau) + \overline{\Phi(\sigma,\tau)}\right)d\tau = \\ = 2\mu^*(u' + iv');\ \Phi(\sigma,\tau) = \phi'(\sigma,t),\ \ \sigma \in L \tag{2}$$

And the first boundary condition of the main problem has the form [3]

$$\varphi(\sigma,t) + \sigma\overline{\varphi'(\sigma,\tau)} + \overline{\psi'(\sigma,\tau)} = i\int_0^\sigma (X_n + iY_n)ds,\ \ \sigma \in L \tag{3}$$

or

$$\Phi(\sigma,t) + \overline{\Phi(\sigma,t)} + \sigma\overline{\Phi'(\sigma,\tau)} + \overline{\Psi'(\sigma,\tau)} = N(\sigma,\tau) + iT(\sigma,t),\ \ \sigma \in L \tag{4}$$

where: $L = L_1 \cup L_0$; $L_1 = U_{k=1}^4 L_k^{(1)}, L_k^{(1)}$ – the sides of the rectangle, L_0 – the boundary of the holes, and by t we will always mean the time parameter.

Taking into account (4), condition (2) can be written in the form:

$$\Gamma\Phi(\sigma,t) - M[N(\sigma,t) + iT(\sigma,t)] = 2\mu^*(u' + iv') \tag{5}$$

Where Γ and M are time operators t:

$$\Gamma\Phi(\sigma,t) = \int_0^t \left[ae^* e^{k(\tau-t)} + 2e^{m(\tau-t)}\right]\Phi(\sigma,\tau)d\tau \tag{6}$$

$$M[N(\sigma,t) + iT(\sigma,t)] = \int_0^t e^{m(\tau-t)}\,[N(\sigma,t) + iT(\sigma,t)]d\tau \tag{7}$$

Given that $(u + iv) = (V_n + iV_\tau)e^{i\alpha(\sigma)}$, ($\alpha(\sigma)$ the angle between the ox axis and the outer normal to the contour L_1 at the point $\sigma \in L_1$), $V_n(\sigma) = \text{const}$, $V_\tau(\sigma) = 0$, $\sigma \in L_1$, $V_n(\sigma) = V_\tau(\sigma) = 0$, $\sigma \in L_0$; $T(\sigma,t) = 0$, $\sigma \in L_1$; $N(\sigma,t) = T(\sigma,t) = 0$, $\sigma \in L_0$, $Re\,\Phi(\sigma,t) = \frac{\sigma_\vartheta(\sigma,t)}{4} = \frac{K_d(t)}{4}$, $\sigma \in L_0$, from (5) we obtain

$$\begin{aligned} &Re\,\Gamma\Phi(\sigma,t) = \Gamma K(t),\ \ \sigma \in L_0 \\ &Im\,\Gamma\Phi(\sigma,t) = 0,\ \ \sigma \in L_1 \end{aligned} \tag{8}$$

where $K(t) = \frac{K_0(t)}{4}$.

From (8) we obtain the Riemann-Hilbert boundary value problem:

$$\begin{aligned} &Re[\Gamma\Phi(\sigma,t) - \Gamma K(t)] = 0,\ \sigma \in L_0; \\ &Im[\Gamma\Phi(\sigma,t) - \Gamma K(t)] = 0,\ \sigma \in L_1. \end{aligned} \tag{9}$$

Let the function $z = \omega(\zeta)$ conformally map the domain S onto a circular ring $D = \{1 < |\zeta| < R\}$ and introduce the notation $l = l_0 \cup l_1$, where $l_0 = \{|\zeta| = 1\}$ and $l_1 = \{|\zeta| = R\}$ are line samples L_0 and L_1 under the mapping $z = \omega(\zeta)$.
From (9) after mapping the area S to D, we obtain the Rienmann-Hilbert boundary value problem for the circular ring D.

$$\begin{aligned} &\mathrm{Re}[\Gamma\Phi_0(\eta,t) - \Gamma K(t)] = 0,\ \eta \in l_0; \\ &\mathrm{Im}[\Gamma\Phi_0(\eta,t) - \Gamma K(t)] = 0,\ \eta \in l_1. \end{aligned} \tag{10}$$

where $\Phi_0(\zeta,t) = \Phi[\omega(\zeta),t]$.

Problem (10) has only a trivial solution, and thus to determine the function, $\Phi_0(\zeta,t)$ we obtain

$$\Gamma[\Phi_0(\zeta,t)-K(t)]=0 \tag{11}$$

It is easy to show that equation (11) has only a trivial solution and, thus for the function, $\Phi(z,t)$ we obtain the formula:

$$\Phi(z,t)=K(t),\ z=S \tag{12}$$

Therefore, for the complex potential $\varphi(z,t)$, taking into account the equality $\varphi'(z,t)=\phi(z,t)$ we will have:

$$\varphi(z,t)=z\cdot K(t) \tag{13}$$

Taking into account the equality $X_n+Y_n=(N+iT)e^{i\alpha(\sigma)}$, and taking into account (13), from (1) and (3) we obtain:

$$e^{-i\alpha(\sigma)}\Gamma[\sigma K(t)]=2\mu^* V_n(\sigma)+MC(\sigma),\ \sigma\in L_1, \Gamma[\sigma K(t)]=0,\ \sigma\in L_0 \tag{14}$$

where

$$C(\sigma)=i\int_0^t N(\zeta_0)e^{i[\alpha(\zeta_0-\alpha(\zeta)]}d\zeta_0=\sum_{j=1}^{r}\int_{L_1^{(j)}}N(\zeta_0)\,sin[\alpha_j-\alpha_r]\,ds_0=C_{r_2}=\text{const}, \sigma\in L_1, r=1{,}4.$$

Boundary condition (14) after mapping the domain S onto D differentiating along the arc abscissa, taking into account the piecewise constancy of the right side of (14), can be written as:

$$\begin{aligned}&\mathrm{Re}\left[e^{-i\alpha}i\eta\Omega(\eta,t)\right]=0,\ \eta\in l_1;\\ &\mathrm{Im}[i\eta\Omega(\eta,t)]=0,\ \eta\in l_0.\end{aligned} \tag{15}$$

where

$$\Omega(\eta,t)=\Gamma[K(t)\omega'(\eta,t)] \tag{16}$$

Consider the function

$$T(\zeta)=\left(1-\frac{1}{\zeta}\right)^2\prod_{j=1}^{\infty}\left(1-\frac{1}{R^{2j}\zeta}\right)^2\cdot\left(1-\frac{\zeta}{R^{2j}}\right)^2 \tag{17}$$

It is easy to show that $T(\zeta)$ we satisfy the condition:

$$\overline{T(\eta)}=\eta^2T(\eta),\ \ \eta\in l_0;\ \ \overline{T(\eta)}=T(\eta),\qquad \eta\in l_1$$

And consequently, the boundary conditions (15) with respect to the function $\chi(\eta,t)=\frac{\Omega(\eta,t)}{T(\eta)}$, can be written in the form:

$$\begin{aligned}&\mathrm{Re}\left[i\eta e^{-i\alpha(\eta)}\chi(\eta,t)\right]=0,\ \eta\in l_1;\\ &\mathrm{Im}[i\chi(\eta,t)]=0,\ \eta\in l_0.\end{aligned} \tag{18}$$

The solvability condition for problem (18) hass the form $\prod_{j=1}^{4}\left(\frac{a_j}{R}\right)^{-\frac{1}{2}}=1$, and the solution of this class problem itself h₀ (for this class, see [3]) is represented by the formula:

$$\chi(\zeta,t)=E(\zeta) \tag{19}$$

Where

$$E(\zeta) = K^0 \cdot \prod_{k=1}^{4}\left(1-\frac{a_k}{\zeta}\right)^{-\frac{1}{2}} \cdot \prod_{j=1}^{\infty}\prod_{k=1}^{4}\left(1-\frac{\zeta}{R^{2j}a_k}\right)^{-\frac{1}{2}} \cdot \left(1-\frac{a_k}{R^{2j}\zeta}\right)^{-\frac{1}{2}} \tag{20}$$

(K^0 – real constant).
Thus, from (16) and (19) we finally obtain:

$$\Gamma[K(t)\omega'(\eta,t)] = T(\zeta)\cdot E(\zeta) \tag{21}$$

where $T(\zeta)$ and $E(\zeta)$ are deifned by formulas (17) and (20), respectively.

Thus, the definition of a conformally mapping function, and thus the definition of the equation of the desired equal-strength contour, is reduced to solving an equation of the Voltaire type (21). Introducing the notation

$$K(t)\omega'(\zeta,t) = \Omega(\zeta,t)\; T(\zeta)E(\zeta) = F_0(\zeta) \tag{22}$$

From (6) and (21) we obtain the equation

$$\int_0^t \left[æ^* e^{k(\tau-t)} + 2e^{m(\tau-t)}\right]\Omega(\zeta,t) = F_0(\zeta) \tag{23}$$

Differentiating (23) with respect to t and adding the resulting equality with (23) multiplied by m, we get:

$$(m-k)æ^* \int_0^l e^{k\tau}\,\Omega(\zeta,\tau)d\tau + (æ^*+2)e^{kt}\Omega(\zeta,t) = mF_0(\zeta)e^{kt} \tag{24}$$

From (24) by differentiation with respect to t we obtain a differential equation of the first kind

$$\dot{\Omega}(\zeta,t) + a\Omega(\zeta,t) = F(\zeta) \tag{25}$$

Where

$$a = \frac{mæ^* + 2k}{æ^* + 2};\; F(\zeta) = \frac{kmF_0(\zeta)}{æ^*+2} \tag{26}$$

(in the expression $\dot{\Omega}(\zeta,t)$ the dot means the derivative with respect to t).
Based on the consideration that in the Kelvin-Voight model, the deformations (hence, the stresses) change exponentially, in the future we will assume that the function $K(t)$ has the form:

$$K(t) = \sigma_0(1+e^{-\varepsilon t}) \tag{27}$$

where σ_0 and ε are positive constants.
The solution of equation (25) has the form

$$\Omega(\zeta,t) = e^{-at}\cdot\left[\Omega(\zeta,o) + \frac{F(\zeta)}{a}\cdot(a^{at}-1)\right] \tag{28}$$

From (22) and (24) we have $\Omega(\zeta,t) = K(o)\omega^{'}(\zeta,o) = KF(\zeta)$ and thus for the conformally mapping function, we finally obtain the formula:

$$\omega'(\zeta,t) = \frac{F(\zeta)}{a\sigma_0(1+e^{-\varepsilon t})}\left[1+(ak-1)\cdot e^{-at}\right] \tag{29}$$

where α and $F(\zeta)$ are defined by formula (26).

After determining $\omega'(\zeta, t)$ the equation of the desired contour, it will be written in the form $\sigma_{\zeta}^{'} = \dfrac{i\eta\omega'(\eta, t)}{|\omega'(\eta, t)|}, \sigma \in L_0, \eta \in l_0$.

4. Conclusion

The condition of equal strength of the desired contour is that the tangential normal stress on it takes a constant value. Note that the mentioned voltage is a function of point and time. To solve the problem, methods of the theory of conformal mappings and boundary value problems of analytic functions are used, and the equation of the desired equal-strength contour is constructed efficiently.

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O problemie znalezienia konturu stałej wytrzymałości wewnątrz lepkosprężystego prostokąta

STRESZCZENIE:
Rozważono problem znalezienia konturu o stałej wytrzymałości wewnątrz lepkosprężystego prostokąta zgodnie z modelem Kelvina-Voigta. Zakłada się, że na boki prostokąta działają stałe normalne siły ściskające o danych wektorach głównych (lub znane są wartości stałych przemieszczeń normalnych), a wewnętrzna granica (żądany kontur o jednakowej wytrzymałości) jest wolna od sił zewnętrznych. Do badania omawianych w artykule problemów zginania płyt wykorzystano metody teorii odbić konforemnych, zagadnienia całki typu Cauchy'ego i wartości brzegowych funkcji analitycznych, które z kolei opierają się na zadaniu skonstruowania funkcji odwzorowującej konformalnie na podwójnie połączonym pierścieniu kołowym ograniczonym linią przerywaną. To ostatnie sprowadza się do problemu Riemanna-Hilberta dla pierścienia kołowego na podstawie rozwiązania, które możliwe jest jako przedstawienie wspomnianej funkcji w postaci wadliwej. Warto zauważyć, że rozpatrując mieszane problemy zginania blachy dla obszarów podwójnie połączonych ograniczonych linią przerywaną, można je rozłożyć na dwa niezależne problemy, z których każdy jest problemem Riemanna-Hilberta.

SŁOWA KLUCZOWE:
model Kelvina-Voigta; wzory Kolosova-Mushelishvili; problemy Riemanna-Hilberta; równanie Volterra